The Compensated Poisson Process

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In Part I of this series we defined the Poisson Process and developed the mathematics to calculate stock price. In Part II of this series we will define and develop the mathematics for the Compenated Poisson Process. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We are tasked with building a model to forecast ABC Company stock price given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
S_0	Stock price at time zero ($)	10.00
ϕ	Expected rate of drift (%)	5.00
ω	Jump size mean (%)	2.50
υ	Jump size volatility (%)	6.00
λ	Average number of annual jumps (#)	4.00
t	Time in years (#)	3.00

Our task is to answer the following questions...

Question 1: What is random stock price at the end of year 3 given that there were k = 10 jumps drawn from a Poisson distribution and y = 0.65 drawn from a normal distribution.

Question 2: What is expected unconditional stock price at the end of year 3?

Building Our Model

In Part One we defined the variable λ to be jump intensity, which is the average number of jumps realized over a given time interval, and the variable k to be the number of jumps realized over the time interval $[0, t]$. The number of jumps is a Poisson-distributed random variable. The equation for the probability of k jumps over the time interval $[0, t]$ is... [3]

$$\mathrm{Prob}\left[k\right] = \frac{(\lambda\, t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\, t\right\} \tag{1}$$

If we are currently standing at time t then the equation for the probability that a jump will arrive over the time interval $[t, t + \delta t]$ is... [2]

$$\mathrm{Prob}\left[\text{Jump arrives over time interval } [t, t+\delta t] \,\middle|\, \text{Currently standing at time } t\right] = \lambda\, \delta t \tag{2}$$

In Part I we defined the variable ϕ to be total return excluding jumps, the variable ω to be jump size mean, and the variable υ to be jump size volatility. In Part I we defined random conditional stock price to be the following equation... [3]

$$S(k)_t = S_0\,\mathrm{Exp}\left\{\phi\, t + k\,\ln(1+\omega) - k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\} \text{ ...where... } y \sim N\left[0, 1\right] \tag{3}$$

Using Equation (3) above the equations for expected conditional and unconditional stock price from Part I are... [3]

$$\mathbb{E}\left[S(k)_t\right] = S_0\,\mathrm{Exp}\left\{\phi\, t + k\,\ln(1+\omega)\right\} \text{ ...and... } \mathbb{E}\left[S_t\right] = S_0\,\mathrm{Exp}\left\{\phi\, t + \omega\,\lambda\, t\right\} \tag{4}$$

For the Compensated Poisson process we want to define the variable μ to be total return including jumps and then negate the expected contribution to total return from the random number of jumps. Note the following equation...

$$\text{if... } \phi = \text{total return excluding jumps } \text{...then... } \mu - \lambda\,\omega = \phi \text{ ...such that... } \mu = \phi + \lambda\,\omega \tag{5}$$

Using the definition in Equation (5) above we can rewrite Equation (3) above as...

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{\mu\,t - \lambda\,\omega\,t + k\,\ln(1+\omega) - k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\} \text{ ...where... } y \sim N\Bigg[0,1\Bigg] \tag{6}$$

Using the definition in Equation (5) above we can rewrite expected conditional stock price Equation (4) above as...

$$\mathbb{E}\Bigg[S(k)_t\Bigg] = S_0 \,\mathrm{Exp}\left\{\mu\,t - \lambda\,\omega\,t + k\,\ln(1+\omega)\right\} = S_0 \,\mathrm{Exp}\left\{\mu\,t - \lambda\,\omega\,t\right\}\Bigg(1+\omega\Bigg)^k \tag{7}$$

Using Equations (1) and (7) above the equation for expected unconditional stock price at time t is...

$$\mathbb{E}\Bigg[S_t\Bigg] = \sum_{k=0}^{\infty} \frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\mathbb{E}\Bigg[S(k)_t\Bigg] \tag{8}$$

Using the equations from Part I the solution to Equation (8) above is... [1]

$$\mathbb{E}\Bigg[S_t\Bigg] = S_0 \,\mathrm{Exp}\left\{\mu\,t - \lambda\,\omega\,t\right\}\mathrm{Exp}\left\{(1+\omega)\,\lambda\,t\right\}\mathrm{Exp}\left\{-\lambda\,t\right\} = S_0 \,\mathrm{Exp}\left\{\mu\,t\right\} \tag{9}$$

The Answers To Our Hypothetical Problem

Using Equation (5) above and the data in Table 1 above the equation for total return under the Compensated Poisson Process is...

$$\mu = \phi + \lambda\,\omega = 0.0500 + 4 \times 0.0250 = 0.1500 \tag{10}$$

Question 1: What is random stock price at the end of year 3 given that there were k = 10 jumps drawn from a Poisson distribution and y = 0.65 drawn from a normal distribution.

Using Equations (6) and (10) above and the data in Table 1 above the answer to the question is...

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{(0.15 - 4 \times 0.025) \times 3 + 10 \times \ln(1 + 0.025) - 10 \times \frac{1}{2} \times 0.06^2 + 0.06 \times \sqrt{10} \times 0.65\right\} = 16.52 \tag{11}$$

Question 2: What is expected unconditional stock price at the end of year 3?

Using Equations (9) and (10) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\Bigg[S_3\Bigg] = 10.00 \times \mathrm{Exp}\left\{0.1500 \times 3\right\} = 15.68 \tag{12}$$

References

[1] Gary Schurman, *The Poisson Process,* March, 2021.

[2] Gary Schurman, *Modeling Exponential Arrival Times,* September, 2015.

[3] Gary Schurman, *The Poisson Distribution,* June, 2012.